The Four Corners Problem Over Larger Alphabets

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Developments in Language Theory (DLT 2026)

Outline

1. Background & Prior Works
2. Corner Pattern Cases
3. Conditional Lower Bounds
4. Upper Bound for Square Case
5. Conclusion

The Four Corners Problem

- We study a 2D combinatorial pattern matching problem.
- Fix an alphabet Σ and a corner pattern $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ where $a, b, c, d \in \Sigma$.
- **Input:** A matrix with entries over Σ.
- **Question:** Does there exist a rectanglar subblock within the input matrix where the upper left corner is a, upper right is b, lower left is c, and lower right is d?
- We denote this problem by $\texttt{Rect-}\Sigma^{m\times n}\texttt{-}\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$.

Corner Pattern Examples

- Consider two examples of corner patterns $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$:

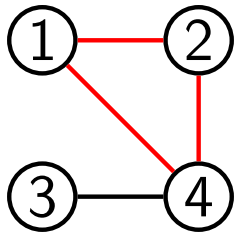
$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Known Upper Bounds

- For any finite alphabet Σ and corner pattern $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, $\texttt{Rect-}\Sigma^{m\times n}\texttt{-}\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is an example of a local picture language (LOC) matching problem.
- We previously showed that all LOC matching problems are solvable in $O(mn \min\{m, n\})$ time (Mráz, Průša, Wehar; CIAA 2019).
- Furthermore, we showed that when $\Sigma = \{0, 1\}$, $\texttt{Rect-}\Sigma^{m\times n}\texttt{-}\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is solvable in $\tilde{O}(mn)$ time for all corner patterns (DLT 2020).
- **Question:** Can we solve $\texttt{Rect-}\Sigma^{m\times n}\texttt{-}\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in $\tilde{O}(mn)$ time when Σ contains more than two symbols?

Triangle Finding

- Given a graph $G = (V, E)$, do there exist vertices v_1, v_2, $v_3 \in V$ such that $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_1\} \in E$?



- Solvable in $O(n^\omega)$ time where $n = |V|$ and $\omega < 2.373$ denotes the matrix multiplication exponent.
- It is unknown if there is an algorithm solving it in $\tilde{O}(n^2)$ time.

Triangle Finding Hardness

- A problem is **Triangle Finding-hard** if it is at least as hard as Triangle Finding.
- In other words, a problem X is Triangle Finding-hard if there exists an efficient reduction from Triangle Finding to X.
- A reduction is a function r that translates graphs to instances of problem X such that G is a graph that contains a triangle if and only if $r(G)$ is a positive instance of problem X.
- Our reductions reduce graphs with n vertices to matrices with $\tilde{O}(n)$ rows and $\tilde{O}(n)$ columns in at most $\tilde{O}(n^2)$ time.

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Corner Patterns Cases

- When $\Sigma = \{0, 1\}$, there are $2^4 = 16$ different possible cases.
- All 16 cases can be reduced to the following 4 cases by simple transformations (rotation, bit-flip): $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$.
- When $\Sigma = \{0, 1, 2\}$, we can reduce down to six distinct cases. The two additional cases are: $\left[\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 & 0 \\ 1 & 2 \end{smallmatrix}\right]$.
- Let $\Sigma = \{0, 1, \ldots, C\}$ for some $C \geq 3$, we can reduce down to seven distinct cases. The one additional case is: $\left[\begin{smallmatrix} 0 & 1 \\ 2 & 3 \end{smallmatrix}\right]$.

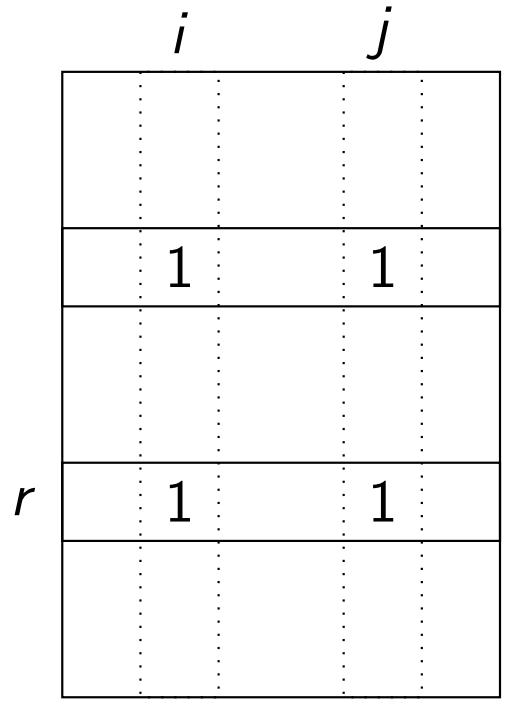
Prior Results

- All four cases over a binary alphabet are solvable in $\tilde{O}(mn)$ time.
  - $\left[\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right]$: Column Pairs Method (we'll see shortly)
  - $\left[\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right]$: Reducing Matches to Minimal Matches (DLT 2020)
  - $\left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$: Divide and Conquer (DLT 2020)
  - $\left[\begin{smallmatrix}1 & 0\\ 1 & 0\end{smallmatrix}\right]$: Column Pairs + Divide and Conquer (DLT 2020)
- One case over ternary alphabet was previously known to be hard.
  - $\left[\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right]$ with $\Sigma = \{0, 1, 2\}$ is Triangle Finding-Hard. (CIAA 2019)

Algorithm for Rect-$\{0,1\}^{m\times n}$-$\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$

Input: $\mathrm{M} \in \{0,1\}^{m,n}$ (assume $m \geq n$)

```
1:  S := ∅
2:  for each row r of M do
3:      R := ∅
4:      for each i s.t. M(r, i) = 1 do
5:          add i to R
6:      end for
7:      for each i, j ∈ R s.t. i < j do
8:          if S contains (i, j) then
9:              return matching submatrix
10:         else
11:             add (i, j) to S
12:         end if
13:     end for
14: end for
```



$$O\left(n^2+mn\right) = O\left(mn\right)$$

New Results

- Cases over alphabet $\Sigma = \{0, 1, 2\}$:
  - $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$: Triangle Finding-Hard (previously known)
  - $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$: Triangle Finding-Hard
  - $\left[\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right]$: Triangle Finding-Hard
  - $\left[\begin{smallmatrix} 0 & 0 \\ 1 & 2 \end{smallmatrix}\right]$: Triangle Finding-Hard
  - $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$: Solvable in $O(mn + n^2)$ time using column pairs method, but cannot be solved in $\tilde{O}(mn)$ time unless 4-Clique is in $\tilde{O}(n^3)$ time!
- Case over alphabet $\Sigma = \{0, 1, 2, 3\}$:
  - $\left[\begin{smallmatrix} 0 & 1 \\ 2 & 3 \end{smallmatrix}\right]$: Triangle Finding-Hard
- Case over alphabet $\Sigma = \{0, 1, \ldots, C\}$ for $C \geq 2$:
  - $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$: Solvable in $O(mn)$ time using column pairs method.

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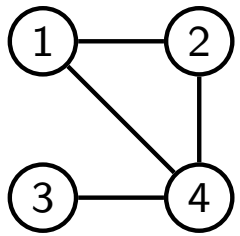
Rect-$\{0,1,2\}^{m\times n}$-$\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$ is Triangle Finding-hard

Theorem

Rect-$\{0,1,2\}^{m\times n}$-$\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$ cannot be solved in $\tilde{O}(mn)$ time unless Triangle Finding can be solved in $\tilde{O}(n^2)$ time.

G:



	1	2	3	4
1	2	1	2	1
2	1	2	2	1
3	2	2	2	1
4	1	1	1	2

$$\left[\begin{array}{cccc|cccc}
2 & \boxed{1} & 2 & 1 & 2 & 0 & 2 & \boxed{0} \\
1 & 2 & 2 & 1 & 0 & 2 & 2 & 0 \\
2 & 2 & 2 & 1 & 2 & 2 & 2 & 0 \\
1 & 1 & 1 & 2 & 0 & 0 & 0 & 2 \\
\hline
2 & 0 & 2 & 0 & 1 & 2 & 2 & 2 \\
0 & 2 & 2 & 0 & 2 & 1 & 2 & 2 \\
2 & 2 & 2 & 0 & 2 & 2 & 1 & 2 \\
0 & \boxed{0} & 0 & 2 & 2 & 2 & 2 & \boxed{1}
\end{array}\right]$$

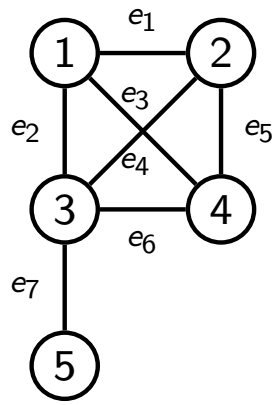
$G = (V, E)$ contains a triangle iff M contains submatrix $\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$.

Rect-$\{0,1,2\}^{m\times n}$-$\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right]$ is 4-Clique-hard

Theorem

Rect-$\{0,1,2\}^{m\times n}$-$\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right]$ cannot be solved in $\tilde{O}(mn)$ time unless 4-Clique is solvable in $\tilde{O}(n^3)$ time.

G:



$$\left[\begin{array}{ccccccc|ccccccc}
1 & \boxed{1} & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & \boxed{0} & 0 & 2 \\
1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 2 \\
2 & \boxed{1} & 2 & 1 & 2 & 1 & 1 & 0 & 2 & 0 & 2 & \boxed{0} & 2 & 2 \\
2 & 2 & 1 & 2 & 1 & 1 & 2 & 0 & 0 & 2 & 0 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{array}\right]$$

dims $= |V| \times 2|E|$

$G = (V, E)$ contains a 4-Clique iff M contains submatrix $\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right]$.

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Square Case

- Fix an alphabet Σ and a corner pattern $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$.
- Given an input matrix M over Σ, we consider whether M contains a subblock with equal width and height matching the corner pattern.
- We denote this problem by $\texttt{Square-}\Sigma^{m\times n}\texttt{-}\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$.
- We can simply solve this problem in $O(mn \min\{m, n\})$ time by iterating over each entry a (as the upper left corner) and iterating over each possible side length to check if it forms a matching square.
- **Question:** Can we do better? Can we solve $\texttt{Square-}\Sigma^{m\times n}\texttt{-}\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in strongly subcubic time?

A Small Algorithmic Improvement: Part 1

Theorem

Square-$\Sigma^{m\times n}$*-*$\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ *is solvable in* $O(\frac{mn\min\{m,n\}}{\sqrt{\log(m+n)}})$ *time.*

- **Basic Idea:** Break your m by n matrix into blocks of size $\mathcal{C}\cdot\sqrt{\log(m+n)}$ by $\mathcal{C}\cdot\sqrt{\log(m+n)}$.
- Now, we have a matrix of $\frac{m}{\mathcal{C}\cdot\sqrt{\log(m+n)}}$ by $\frac{n}{\mathcal{C}\cdot\sqrt{\log(m+n)}}$ blocks.
- If there is a square matching the corner pattern, then there is a combination of at most 4 blocks containing the corners of the square.

A Small Algorithmic Improvement: Part 2

Theorem

Square-$\Sigma^{m\times n}$*-*$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is solvable in* $O(\frac{mn\min\{m,n\}}{\sqrt{\log(m+n)}})$ *time.*

- There are at most $\frac{mn\min\{m,n\}}{\mathcal{C}^3\cdot\log^{1.5}(m+n)}$ combinations to consider.
- However, there are at most $|\Sigma|^{4\cdot\mathcal{C}^2\cdot\log(m+n)}$ possible combinations so some will appear multiple times.
- Let us cache the results so that we don't need to redo work.
- If we pick $\mathcal{C}$ sufficiently small relative to the alphabet size, then we can precompute all of these combination cases in $\tilde{O}(mn)$ time storing the results in a table.
- Lookups in the table will take $O(\log(m+n))$ time. As a result, checking all cases will take $O(\frac{mn\min\{m,n\}}{\sqrt{\log(m+n)}})$ time.

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Conclusion

We showed the following:

- Five cases of the four corners problem over larger than binary alphabets are Triangle Finding-Hard.
- One particular case (namely, $\left[\begin{smallmatrix}1 & 0\\ 1 & 0\end{smallmatrix}\right]$) is solvable in $O(mn + n^2)$ time, but is not solvable in $\tilde{O}(mn)$ time unless 4-Clique is in $\tilde{O}(n^3)$ time.
- All square cases are solvable slightly faster than cubic time.

Thank You!

Questions or Comments?