Intersection Non-Emptiness and Hardness within Polynomial Time

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Overview

1. Background
2. Our Contribution
3. Unary Finite Automata
4. Main Result

Intersection Non-Emptiness Problem

Intersection Non-Emptiness for DFA's

Given a finite list of DFA's, is there a string that is accepted all of the DFA's?

- This is a classic PSPACE-complete problem [Kozen 1977].
- Standard solution considers the state diagram of the Cartesian product automaton.
- For k DFA's with n states each, we can solve this problem in $O(n^k)$ time.

It's Hard to Find Faster Algorithms

- **More recent work:** If we can solve intersection non-emptiness in $n^{o(k)}$ time, then
  - there exist faster algorithms for several hard problems[1]
  - the complexity classes NL and P are not equal[2]
- If for some $k \in \mathbb{N}$ and $\varepsilon > 0$ we can solve intersection non-emptiness for k DFA's in $O(n^{k-\varepsilon})$ time, then
  - there exist slightly faster algorithms for SAT and QBF[3]
- Although these results are more recent, conditional lower bounds for intersection non-emptiness with a fixed number of DFA's go back to Kasai and Iwata 1985.

[1] Several papers including KLV 2003 and Fernau and Krebs 2016
[2] Wehar 2014
[3] Wehar 2016

Triangle Finding and 3SUM

- **Triangle Finding:** Given a Graph G with n vertices, do there exist vertices a, b, and c such that a and b are adjacent, b and c are adjacent, and c and a are adjacent?
  - Solvable in $O(n^{2.373})$ time by a reduction to matrix multiplication
- **3SUM:** Given a set of integers S of size n, do there exist elements a, b, and c such that $a + b + c = 0$?
  - Solvable in $O(n^2)$ time and sublogarithmic factor improvements are known
- It is considered hard to find faster algorithms for Triangle Finding and 3SUM.

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Direct Reductions

- We investigate the intersection non-emptiness problem when $k = 2$ and $k = 3$.
- We provide direct reductions:
  - From Triangle Finding for a graph with n vertices and m edges to intersection non-emptiness for two DFA's where the first DFA has $O(m \log(n))$ states and the second DFA has $O(n \log(n))$ states.
  - From 3SUM for a set of n integers in $[-n^k, n^k]$ to non-emptiness of intersection for three DFA's where each DFA has $O(kn \log(n))$ states.

Faster Algorithms

- We show that there exist faster algorithms when the DFA's are restricted to a unary input alphabet.
- Intersection non-emptiness for two unary DFA's is solvable in linear time.
- We connect the complexity of intersection non-emptiess for three unary DFA's to the complexity of Triangle Finding:
  - **Main Result:** For every $\alpha > 0$, intersection non-emptiness for three unary DFA's is solvable in $O(n^{\frac{\alpha}{2}})$ time if and only if Triangle Finding is solvable in $O(n^{\alpha})$ time.
  - Since Triangle Finding can be solved in $O(n^{2.373})$ time, intersection non-emptiness for three unary DFA's can be solved in $O(n^{1.1865})$ time.

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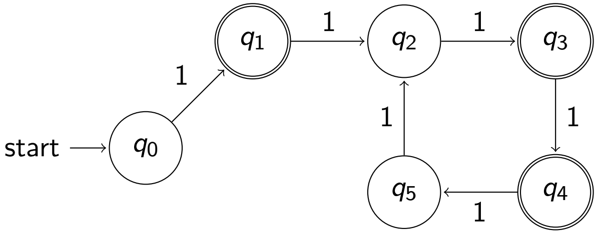
Unary Finite Automata

- A unary DFA is simply a DFA over a unary input alphabet.
- Such automata consist of a segment and a cycle.
- Each final state from a unary DFA can be viewed as a constraint over natural numbers.
- It is known that intersection non-emptiness for unary DFA's is NP-complete [Stockmeyer and Meyer 1973].

Unary DFA: Example



- The segment has length two and the cycle has length four.
- The DFA accepts:
  - the string of length 1
  - strings with length congruent to 3 mod 4
  - strings with length greater than 0 and divisible by 4

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Main Result: Part 1

Theorem

If intersection non-emptiness for three unary DFA's $\in DTIME(n^{\frac{\alpha}{2}})$, then Triangle Finding $\in DTIME(n^{\alpha})$.

- Given a graph G with n vertices, we construct DFA's D_1, D_2, and D_3 with $O(n^2)$ states each such that G contains a triangle if and only if $L(D_1) \cap L(D_2) \cap L(D_3) \neq \emptyset$.
- We can compute three relatively prime integers a_1, a_2, and $a_3 \in [n, 5n]$ such that:
  - D_1 will have cycle length $a_1 \cdot a_2$
  - D_2 will have cycle length $a_2 \cdot a_3$
  - D_3 will have cycle length $a_3 \cdot a_1$

Main Result: Part 1

Theorem

If intersection non-emptiness for three unary DFA's $\in DTIME(n^{\frac{\alpha}{2}})$, then Triangle Finding $\in DTIME(n^{\alpha})$.

- The DFA's read in an integer that is uniquely associated with a triple of integers (i, j, k).
- This association is a bijective mapping from triples of integers in $[0, a_1) \times [0, a_2) \times [0, a_3)$ to integers in $[0, a_1 \cdot a_2 \cdot a_3)$:
  - (i, j, k) maps to $i \cdot a_2 \cdot a_3 + j \cdot a_3 \cdot a_1 + k \cdot a_1 \cdot a_2 \ mod \ a_1 \cdot a_2 \cdot a_3$.

Main Result: Part 1

Theorem

If intersection non-emptiness for three unary DFA's $\in DTIME(n^{\frac{\alpha}{2}})$, then Triangle Finding $\in DTIME(n^{\alpha})$.

- The integers i, j, and k each represent a choice of a vertex from G when i, j, and $k \in [0, n)$, respectively.
- If an input encodes a triple of integers (i, j, k), then
  - D_1 accepts when (i, j) represents an edge in G
  - D_2 accepts when (j, k) represents an edge in G
  - D_3 accepts when (k, i) represents an edge in G
- Together the DFA's collectively determine whether or not an input encodes a triangle.

Main Result: Part 2

Theorem

If Triangle Finding $\in DTIME(n^{\alpha})$, then intersection non-emptiness for three unary DFA's $\in DTIME(n^{\frac{\alpha}{2}})$.

- Let unary DFA's D_1, D_2, and D_3 each with at most n states be given.
- We first consider final states on the segments of the DFA's.
- We can quickly check to see if any of these finals states correspond with a string in the intersection.

Main Result: Part 2

Theorem

If Triangle Finding $\in DTIME(n^{\alpha})$, then intersection non-emptiness for three unary DFA's $\in DTIME(n^{\frac{\alpha}{2}})$.

- Next, we consider final states on the cycles of the DFA's.
- Each final state has an associated natural number constraint that checks the remainder modulo the cycle length.
- Let c_1, c_2, and c_3 denote the cycle lengths of D_1, D_2, and D_3, respectively.
- Let d denote the greatest common divisor of c_1, c_2, and c_3.
- Further, let $c_1' = \frac{c_1}{d}$, $c_2' = \frac{c_2}{d}$, and $c_3' = \frac{c_3}{d}$.

Main Result: Part 2

Theorem

If Triangle Finding $\in DTIME(n^{\alpha})$, then intersection non-emptiness for three unary DFA's $\in DTIME(n^{\frac{\alpha}{2}})$.

- For each integer $i \in [0, d)$, we construct a tripartite graph G_i.
- We construct the G_i's so that the DFA's have a non-empty intersection if and only if there exists $i \in [0, d)$ such that G_i has a triangle.
- Let $i \in [0, d)$ be given. The graph G_i has three groups of vertices such that:
    - the first group has g_1 vertices where $g_1 = gcd(c'_1, c'_2)$
    - the second group has g_2 vertices where $g_2 = gcd(c'_2, c'_3)$
    - the third group has g_3 vertices where $g_3 = gcd(c'_3, c'_1)$

Main Result: Part 2

Theorem

If Triangle Finding $\in DTIME(n^{\alpha})$, then intersection non-emptiness for three unary DFA's $\in DTIME(n^{\frac{\alpha}{2}})$.

- For each $j \in \{1, 2, 3\}$, the vertices in the jth group are labeled with integers from 0 to $g_j - 1$.
- Each edge is associated with a final state from the DFA's.
- There is an edge between the vertex labeled x from the first group and the vertex labeled y from the second group if there is a final state from D_2 whose natural number constraint's remainder is congruent to i mod d, x mod g_1, and y mod g_2.
- Edges are defined similarly between other groups of vertices.

Main Result: Part 2

Theorem

If Triangle Finding $\in DTIME(n^{\alpha})$, then intersection non-emptiness for three unary DFA's $\in DTIME(n^{\frac{\alpha}{2}})$.

- At this point, it can be seen that a triangle of G_i is exactly a choice of finals states f_1, f_2, and f_3 from D_1, D_2, and D_3, respectively, such that:
  - there exists a string that leads to all three finals states
  - each final state's associated remainder is congruent to i mod d

Main Result: Part 2

Theorem

If Triangle Finding $\in DTIME(n^{\alpha})$, then intersection non-emptiness for three unary DFA's $\in DTIME(n^{\frac{\alpha}{2}})$.

- The groups of vertices may be lopsided in the sense that one group may contain more vertices than another.
- We can reduce triangle finding for G_i to triangle finding for a collection of tripartite subgraphs with an equal number of vertices in each group.
- Finally, we just need to solve Triangle Finding on each of these subgraphs.

Main Result: Part 2

Theorem

If Triangle Finding $\in DTIME(n^{\alpha})$, then intersection non-emptiness for three unary DFA's $\in DTIME(n^{\frac{\alpha}{2}})$.

- Based on the definitions of g_1, g_2, and g_3, we have the following inequalities:
  - $g_1 \cdot g_2 \leq c_2'$
  - $g_2 \cdot g_3 \leq c_3'$
  - $g_3 \cdot g_1 \leq c_1'$
- Without loss of generality, the second group is the smallest meaning that $g_2 = min(g_1, g_2, g_3)$.

Main Result: Part 2

Theorem

If Triangle Finding $\in DTIME(n^{\alpha})$, then intersection non-emptiness for three unary DFA's $\in DTIME(n^{\frac{\alpha}{2}})$.

- Therefore, when we rebalance, we can split into roughly $\frac{g_1 \cdot g_3}{g_2^2}$ subgraphs each with $O(g_2)$ states.
- Finally, we get that solving Triangle Finding for all of the balanced subgraphs across all of the G_i's takes time $O(d \cdot g_1 \cdot g_3 \cdot g_2^{\alpha-2}) \leq O(d \cdot {c'_1}^{\frac{\alpha}{2}}) \leq O(d \cdot (\frac{n}{d})^{\frac{\alpha}{2}})$.

Summary of Reduction

- The reduction can be broken into two phases.
- **Phase 1:** We reduce one instance of intersection non-emptiness for three unary DFA's to d instances of triangle finding where each graph has $g_1 + g_2 + g_3$ vertices.
- **Phase 2:** We further reduce each of these instances of triangle finding to $\frac{g_1 \cdot g_3}{g_2^2}$ instances of triangle finding where each graph has $O(g_2)$ vertices.